ABSTRACT

Compact stars are formed when stars run out of fuel to maintain hydrostatic equilibrium through thermal pressure. Fermionic compact stars maintain equilibrium due to the degeneracy pressure produced by fermi gases. The Lane-Emden equation is solved by assuming a polytropic equation of state. For different polytropic indices n, stable and unstable solutions are obtained for the stellar system. The Buchdahl limit is calculated, giving a general relativistic lower bound on the radius of the star, corresponding to where the central pressure diverges and gravity collapses the star. Investigations are made on the effects of fermion mass m_f and strong two-body interactions on the mass-radius relatiosnhips of fermionic compact stars. All degenerate stars have a maximum mass limit that they can support before gravitational collapse wins over. In the non-interacting case, the maximum mass M_{max} and corresponding radius R_{crit} are related to the fermion mass m_f as $M_{max} \propto 1/m_f^2$ and $R_{crit} \propto 1/m_f^2$. The degeneracy pressure can be further enhanced by introducing repulsive two-body interactions in the system. Interactions are mediated by vector bosons of mass m_i. Strong interactions are introduced through m_i = 100 MeV. Interactions further stabilise the star against gravitational collapse, and thus M_{max} increases. For a fixed m_f, the limiting mass and radius are related to the mass of the mediator particle m_i as $M_{max} \propto 1/m_i$ and $R_{crit} \propto 1/m_i$.

Keywords: Fermionic Compact Stars; Degeneracy Pressure; Hydrostatic Equilibrium; Tolman-Oppenheimer-Volkoff Equation; Interacting Fermi Gas; Buchdahl limit; Lane-Emden Equation

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Chapter 1

INTRODUCTION

1.1 What are Compact Stars?

The term *Compact Star* refers to the class of astrophysical objects encompassing White Dwarfs, Neutron Stars and Black Holes. These are the final stages of Stellar Evolution [1, 2].

The life cycle of a star is dictated by a constant balance of two forces. The force of gravity seeking to collapse the star inwards is balanced by the internal pressure produced by the star acting outwards. This is the idea of *Hydrostatic Equilibrium*.

In ordinary Main Sequence stars (such as our Sun), this internal pressure is provided for by the nuclear reactions taking place at its core. It is the thermal pressure that is produced by the nuclear burning of Hydrogen that balances the force of gravity.

Once the Hydrogen at the star's core runs out, the internal pressure produced stops. The star shrinks as gravity wins momentarily, increasing the pressure at the core. If the star is massive enough, this increase in pressure can be sufficient enough to start further stages of nuclear burning. A star can continue to fuse heavier and heavier elements until Iron, beyond which fusion is not possible [1].

The end of a star's life is determined by when this sort of "normal" stellar evolution is no longer feasible. The "death" of a star is seen in its collapse into an extremely dense state. A *Compact Star* is thus formed. These can be White Dwarfs, Neutron Stars or Black Holes, depending on the initial mass of the star in question.

For stars of mass less than 8 solar masses, the mass is insufficient to ignite fusion of elements beyond Carbon. However, the temperature is high enough that the star sheds its outer layers. What is left behind is the **White Dwarf**, a compact remnant that is composed mainly of Carbon and Oxygen. White Dwarfs are held together by *Electron Degeneracy Pressure*. They contain a mass roughly equal to our Sun, in a radius comparable to that of Earth (of the order of 1000 km) [3].

For stars of mass greater than 8 solar masses, the sequence of nuclear burning continues till an iron-rich core is formed. Gravitational collapse wins over the electron degeneracy pressure, and the star becomes more and more compact. High temperatures (of the order of 10^9 K) are reached. As the star reaches higher densities, electrons and protons combine to form neutrons via neutron capture. Ultimately, it is a combination of *Neutron Degeneracy Pressure* and *Strong Interactions* that halts the collapse. The outer envelope of the star is ejected by the flux of neutrinos produced during electron capture, resulting in a supernova explosion and leaving behind a remnant [2, 3].

The division between **Neutron Stars** and **Black Holes** is not a clear-cut one. If the mass of the remnant is below 3 solar masses, we are left with a Neutron Star. If the mass of the remnant left behind after the supernova explosion exceeds this limit, the star collapses into a Black Hole. Black Holes are formed when all the aforementioned stabilisation mechanisms fail in the face of overwhelming gravity. Not even light can escape the immense gravity of such objects.

1.2 Project Objective and Overview

The stabilisation mechanism of Compact Stars is very different when compared to ordinary (Main Sequence) stars. One of the ways we can model a Compact Star is by considering it to be made of a free (i.e., non-interacting) Fermi Gas. The star is then stabilised against gravity due to the Fermi Pressure exerted by the gas.

We also add interactions to the system. The simplest interaction is the two-body repulsion. Repulsive Interactions further stabilise the star against gravitational collapse.

The Mass-Radius relationship of Compact Stars is dependent on the mass of the fermion making up the fermion gas, as well as any interactions that are present within the gas. Specifically - there is a maximum mass that can be stabilised against gravity by the degeneracy pressure exerted by a fermi gas. This macroscopic quantity depends on the two above-mentioned microscopic factors.

In this project, we construct Compact Stars from fermi gases of varying fermion masses. For each mass, we consider the strongly interacting and non-interacting Equations of

State (EoS). By numerically solving for the different cases, the exact effect of fermion mass and interactions on the mass-radius relationship of non-rotating fermionic Compact Stars is analyzed.

All code used in the project was written in Python 3.7.6.

Chapter 2

HYDROSTATIC EQUILIBRIUM

In any given star, there are two competing forces. There is the force of gravity pushing in on the star, balanced by the outward pressure produced by the star. This is *Stellar Hydrostatic Equilibrium*.

2.1 Newtonian Hydrostatic Equilibrium

Stars are modelled as spherically symmetric objects. At every r, there is a balance between the Internal Pressure $P(r)$ as it supports the overlaying material against gravitational attraction of mass energy interior to r.

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{2.1}$$

where $m(r)$ is the mass and $\rho(r)$ is the density of stellar material lying within r.

The mass continuity equation follows directly from spherical symmetry.

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r) \tag{2.2}$$

Equations (2.1) and (2.2) together constitute the *Equations of Stellar Structure*.

2.1.1 Derivation of Equation (2.1)

Consider an infinitesimal volume element of cross-sectional area δs and height δr at a distance of r from the center. Thus, the mass of the infinitesimal element δm,

$$\delta m = \rho(r)\delta s\delta r$$

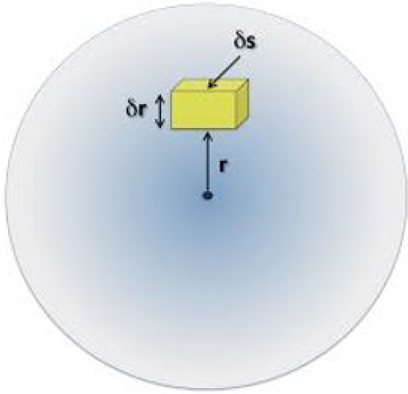


Figure 2.1: Infinitesimal Volume Element in Hydrostatic Equilibrium
(*Ref: Introduction to Astrophysics, C. Bertulani, Texas A&M-Commerce*)

In hydrostatic equilibrium, both outward and inward forces balance [4].

$$P(r)\delta s = P(r+\delta r)\delta s + \frac{Gm(r)\delta m}{r^2}$$
$$P(r)\delta s = P(r+\delta r)\delta s + \frac{Gm(r)\rho(r)\delta r\delta s}{r^2}$$
$$\frac{P(r+\delta r) - P(r)}{\delta r} = -\frac{Gm(r)\rho(r)}{r^2}$$

For an infinitesimal element,

$$\frac{P(r+\delta r) - P(r)}{\delta r} = \frac{dP}{dr}$$

Thus we obtain the equation of hydrostatic equilibrium as

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

2.1.2 Exact Solution: Incompressible Fluid Model

In general, the equations of Stellar Structure - eqn (2.1) and eqn (2.2) - cannot be solved analytically until a relation between P and ρ (EoS) is specified. However, by assuming the special case of an Incompressible Fluid, i.e., a *Constant Density Star*, we obtain an exactly solvable system.

The Density of the star is taken as

$$\rho(r) = \rho_c = \frac{3M}{4\pi R^3}$$

After integrating the equations of stellar structure, the pressure for a star of total mass M and total radius R is

$$P(r) = \frac{3}{8\pi}\frac{GM^2}{R^6}(R^2 - r^2) \tag{2.3}$$

We obtain the central pressure by setting $r = 0$,

$$P_c = \frac{3}{8\pi}\frac{GM^2}{R^4} \tag{2.4}$$

We note that $P_c \to \infty \; as \; R \to 0$. This means that the central gravitational pressure becomes infinite when the star collapses to a singularity (zero radius).

2.2 General Relativistic Corrections

Compact Stars are objects of extremely high density. In the limit of $\frac{2GM}{Rc^2} \to 1$, general relativistic (GR) effects become important.

The critical radius

$$R_s = \frac{2GM}{Rc^2} \tag{2.5}$$

is known as the *Schwarzschild Radius*. This is the radius at which the escape velocity of any object equals the speed of light in vacuum [3].

We describe Compact Stars using Einstein's equation,

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.6}$$

Applying GR corrections to (2.1) gives us the **Tolman-Oppenheimer-Volkoff (TOV) equation**,

$$\frac{dP}{dr} = -\frac{G\epsilon(r)m(r)}{c^2r^2}\left[1 + \frac{p(r)}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 p(r)}{m(r)c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1} \tag{2.7}$$

The TOV equation holds for general relativistic, isotropic fluid spheres in hydrostatic equilibrium.

The mass continuity equation (2.2) is modified as

$$\frac{dm}{dr} = \frac{4\pi r^2 \epsilon(r)}{c^2} \tag{2.8}$$

where $\epsilon(r) = \rho(r)c^2$ is the energy density at a given r.

The GR corrections introduce three new terms to the Hydrostatic Equilibrium equation. All three terms are positive and greater than one. They strengthen the gravity at a given r compared to the Newtonian case [5].

Equations (2.7) and (2.8) together constitute the *Relativistic Equations of Stellar Structure*.

2.2.1 Exact Solution: Schwarzschild Interior Solution

Similar to the Newtonian case, the Relativistic Equations of Stellar Structure are not exactly solvable in general unless the EoS is known. By assuming the Incompressible Fluid model, we obtain a Constant Density star, i.e, a rigid body star. The system is then analytically solvable [6].

We assume that the density $\rho(r)$ and hence the energy density $\epsilon(r)$ is a constant.

$$\epsilon(r) = \epsilon_0 = \rho_0 c^2$$

After integrating the relativistic equations of stellar structure, the pressure for a star of total mass M and total radius R is

$$P(r) = \epsilon_0 \left[\frac{\sqrt{1 - \frac{2GM}{Rc^2}} - \sqrt{1 - \frac{2GMr^2}{R^3c^2}}}{\sqrt{1 - \frac{2GMr^2}{R^3c^2}} - 3\sqrt{1 - \frac{2GM}{Rc^2}}} \right] \tag{2.9}$$

We obtain the central pressure by setting $r = 0$,

$$P_c = \epsilon_0 \left[\frac{\sqrt{1 - \frac{2GM}{Rc^2}} - 1}{1 - 3\sqrt{1 - \frac{2GM}{Rc^2}}} \right] \tag{2.10}$$

2.3 Buchdahl's Limit

We obtain limits on the stability of gravitationally bound systems by noting where the central pressure diverges in the incompressible fluid model.

In the Newtonian case, the central pressure $P_c \to \infty \; as \; R \to 0$. Thus, the central gravitational pressure diverges when the radius goes to zero (point mass).

In the General Relativistic case, $P_c \to \infty \; as \; \frac{GM}{Rc^2} \to \frac{4}{9}$. This gives the condition for stability as $\frac{GM}{Rc^2} < \frac{4}{9}$. This is called *Buchdahl's Limit*. We see that P_c diverges when the radius $R = \frac{9GM}{4c^2} = \frac{9}{8}R_s$ and there is gravitational collapse of the star.

All realistic stars have average energy densities less than a rigid body star of the same central density. Thus, the Buchdahl limit can be extended to all static fluid spheres of non-negative density and pressure under the assumption of energy density not increasing outwards. This extension of Buchdahl limit to all reasonable stellar equations of state is *Buchdahl's theorem*.

The *Compactness* of a star is defined as the ratio of gravitational mass to the radius of the star.

$$C = \frac{GM}{Rc^2} \tag{2.11}$$

Buchdahl Limit in terms of C : All static fluid spheres where the energy density is not increasing outward cannot be more compact than [6]

$$C_{max} = \frac{4}{9}$$

Chapter 3

FERMIONIC COMPACT STARS

3.1 Degenerate Fermi Gases

Fermions are particles with half-integer spin. They follow *Fermi Dirac statistics*. Fermions have anti-symmetric wave functions, and as a consequence, obey the *Pauli Exclusion Principle*. Two identical fermions cannot occupy the same state. This is responsible for the *Fermi Pressure* exerted by a Fermion gas.

At low temperatures and high densities, a fermion gas is said to be *degenerate*. In a degenerate gas, the length scale between the particles of the gas is much smaller than the quantum wavelength of the particles, i.e., wave nature cannot be ignored. In other words, for a degenerate gas, the parameter

$$n\lambda^3 >> 1$$

where n is the number density of the gas, and λ is the mean thermal wavelength of the particles ($\lambda = \frac{h}{\sqrt{2\pi m k_b T}}$) [7].

The Fermi-Dirac distribution function is described by,

$$f(E) = \frac{1}{exp[(E-\mu)/k_b T] + 1}$$

In the completely degenerate case, i.e, at $T = 0\ K$, the distribution reduces to a step function, as all levels below the *Fermi Energy*, E_F are occupied and all levels above E_F are unoccupied.

$$f(E) = \begin{cases} 1 & \text{for } E \leq E_F \\ 0 & \text{for } E > E_F \end{cases}$$

In the completely degenerate case, the pressure of the fermi gas is given by,

$$P(k_F) = \frac{\epsilon_0}{24}\left[(2x^3 - 3x)(1+x^2)^{1/2} + 3\sinh^{-1}(x)\right] \tag{3.1}$$

and the energy density of the gas is given by,

$$\epsilon(k_F) = \frac{\epsilon_0}{8}\left[(2x^3 + x)(1+x^2)^{1/2} - \sinh^{-1}(x)\right] \tag{3.2}$$

where $\epsilon_0 = \frac{m_f^4 c^5}{\pi^2 \hbar^3}$, m_f is mass of the fermion making up the gas, $\hbar$ is the reduced Planck constant, c is the speed of light and $x = \frac{k_F}{m_f c}$, where k_F is the Fermi momentum [5].

3.2 Maximum Mass of Fermionic Compact Stars

Compact Stars obtain hydrostatic equilibrium very differently to regular, main-sequence stars. In the case of free fermion compact stars, the force that pushes against gravitational collapse is the Degeneracy Pressure generated by the fermion gas. In the case of interacting fermions, the repulsive interaction also pushes outwards, providing further stabilisation against gravitational collapse.

This stabilisation occurs until the fermi velocity of the constituent fermions reaches the speed of light. Thus, there exists a maximum mass and a corresponding radius for all compact stars.

3.2.1 Landau's Argument

The concept of a mass limit for degenerate fermion stars was first seen in White Dwarf stars - this is the famous *Chandrasekhar Limit* that won Subrahmanyan Chandrasekhar his Nobel Prize. Before Chandrasekhar computed the limit in 1930, similar calculations combining degeneracy and relativity were established by Wilhelm Anderson and E. C. Stoner in 1929 [8, 9].

In 1932, Lev Landau put forth an argument for the existence of a mass limit for the general degenerate star. This applied to both white dwarfs and neutron stars [10, 11].

Consider a star of radius R, made of N fermions. Number density of fermions is then $n \sim N/R^3$. The volume per fermion is then $\sim 1/n$ which fixes their momentum as $\sim \hbar n^{1/3}$ through the Heisenberg uncertainty principle.

The Fermi energy of a gas particle in the extremely relativistic regime, where the fermion rest mass can be ignored, is

$$E_F \sim \hbar n^{1/3} c \sim \frac{\hbar c N^{1/3}}{R}$$

The gravitational energy per fermion is

$$E_G \sim -\frac{G M m_f}{R}$$

where $M = n m_f$.

Equilibirum is obtained at a minimum of total energy E where

$$E = E_F + E_G = \frac{\hbar c N^{1/3}}{R} - \frac{G N m_f^2}{R} \tag{3.3}$$

Both terms scale as $1/R$. The total energy is dependent only on the number of fermions N in the sphere of a fixed fermion mass m_f. For large N, it is the gravitational energy that dominates while for smaller values of N, the kinetic energy dominates. There are two cases to be considered depending on the number of fermions N in the star [3, 6].

- $E > 0$: This is true for small N. E can be decreased by increasing R. This decreases E_F and the electrons tend to become non-relativistic with $E_F \sim p_F^2 \sim 1/R^2$. This results in E_G dominating over E_F with increasing R so that E becomes negative, increasing to zero as $E \to \infty$. There is stable equilibrium at finite R.
- $E < 0$: E can be decreased without bound by decreasing R - no equilibrium exists and gravitational collapse sets in.

The maximum baryon number at equilibrium is determined by setting $E = 0$ in eqn (3.3). Then,

$$N_{max} \sim \left[\frac{\hbar c}{G m_f^2}\right]^{3/2} \tag{3.4}$$

$$M_{max} \sim N_{max} m_f \tag{3.5}$$

Apart from numerical factors relating to the composition of the star, we obtain a maximum mass that depends only on the fundamental constants. This is the maximum mass for a degenerate star made of fermions of mass m_f.

3.3 Equations of State

The equations of stellar structure is a system of two equations in three variables. This is true in both the Newtonian and GR cases. There is no equation for $\epsilon(r)$ or equivalently, $\rho(r)$. Thus, we require an extra condition (either an equation or an assumption) to make the system solvable.

One way to get around this is to assume a density profile. This was done in the Incompressible Fluid Model where we assumed a constant density. However, this is not possible in general, as a rigid body does not exist in relativity.

Another approach is to find the dependence of ϵ on r implicitly through an equation of state (EoS). The EoS is a relation between the pressure and energy density, i.e, a relation of the form $\epsilon(P)$, allowing us to find $\epsilon(r)$ from $P(r)$. The TOV and mass continuity equations are thus solvable by introducing an additional equation, i.e, the EoS.

3.3.1 The Lane-Emden Equation

An EoS that relates the pressure and density through a power law is called a *Polytropic equation of state.* Polytropes are useful as they provide simpler solutions and are easier to manipulate than other solutions. However, they carry the underlying assumption that the same power law, defined by a constant, holds throughout the entirety of the star [3, 6].

A polytrope defines the gas pressure as $P = K\epsilon^{\gamma} = K\epsilon^{(n+1)/n}$. Here γ is the adiabatic index, and n is the polytropic index. K is a constant. The requirement for an additional equation to solve the equations of stellar structure is thus satisfied [5].

The set of three differential equations in the Newtonian case can now be reduced to a single differential equation whose terms depend only on n and K and this equation can be solved. This is the *Lane-Emden Equation*.

The constants we start with are K and n, along with the central pressure P_c and the central density ϵ_c.

The Newtonian equations of Stellar Structure are

$$\begin{aligned}\frac{dP(r)}{dr} &= -\frac{Gm(r)\rho(r)}{r^2} \\ \frac{dm(r)}{dr} &= 4\pi r^2 \rho(r)\end{aligned}$$

We combine both equations in the following manner,

$$\begin{aligned}\frac{d}{dr}\left[\frac{1}{\rho}\frac{dP}{dr}\right] &= \frac{d}{dr}\left[\frac{-Gm(r)}{r^2}\right] \\ &= \frac{2Gm}{r^3} - \frac{G}{r^2}\frac{dm}{dr} \\ &= \frac{2}{\rho r}\frac{dP}{dr} - 4\pi G\rho\end{aligned}$$

This can be simplified to give

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho$$

We substitute the Polytopic EoS $P = K\rho^{\gamma} = K\rho^{(n+1)/n}$.

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2 K}{\rho}\gamma\rho^{\gamma-1}\frac{d\rho}{dr}\right] = -4\pi G\rho$$

The density is rescaled in form of dimensionless density variable θ as

$$\begin{aligned}\rho &= \rho_c\theta^n \\ P &= K\,\rho_c^{(n+1)/n}\,\theta^{n+1}\end{aligned}$$

Substituting the values of ρ gives us

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\left[r^2\,K\rho_c^{(n+1)/n}(n+1)\,\frac{d\theta}{dr}\right] &= -4\pi G\rho_c\theta^n \\ \frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2\,K\,(n+1)\rho_c^{1/n-1}}{4\pi G}\,\frac{d\theta}{dr}\right] &= -\theta^n\end{aligned}$$

The radius is also cast in dimensionless form by defining α such that

$$\alpha^2 = \frac{K\,(n+1)\,\rho_c^{1/n-1}}{4\pi G}$$

Then, $r = \alpha\xi$, where ξ is the dimensionless radius.

Substituting ξ in place of r gives us

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] + \theta^n = 0 \tag{3.6}$$

This is the Lane-Emden Equation - a dimensionless equation for the gravitational potential of a Newtonian, self-gravitating, spherically symmetric, polytropic fluid.

For a given polytropic index n, we obtain the solution to the Lane-Emden equation as $\theta_n(\xi)$. In general, solutions are obtained numerically by applying the initial conditions $\theta(0) = 1$ and $\theta'(0) = 0$. The first zero of $\theta(\xi)$ returns a value proportional to the radius of the star [3].

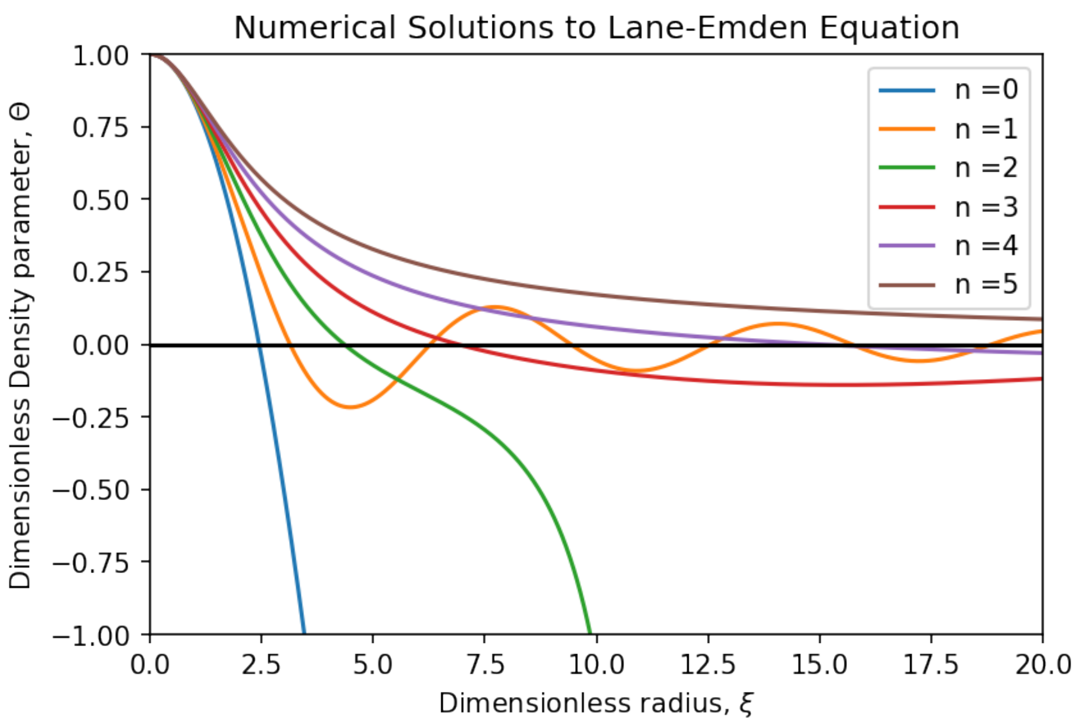


Figure 3.1: Solutions of Lane-Emden equation for different n
Generated using Python 3.7.6

There are three analytic solutions to the Lane-Emden equation corresponding to $n = 0, 1, 5$.

- $n = 0$ corresponds to $\gamma \to \infty$, which describes an Incompressible Fluid model star of constant density.
- $n = 1$ corresponds to $\gamma = 2$, which describes an interaction-dominated EoS.
- In the case $n = 5$, the first zero occurs at ∞. This is the limiting case of a finite mass contained within an infinite radius.
- For $n > 5$ the mass increases with the radius without any bound [6].

3.4 Mass-Radius Relationship of Fermionic Compact Stars

3.4.1 "Star-Maker": The Algorithm

The following algorithm was used to "make" stars, i.e, solve the Relativistic Equations of Stellar Structure.

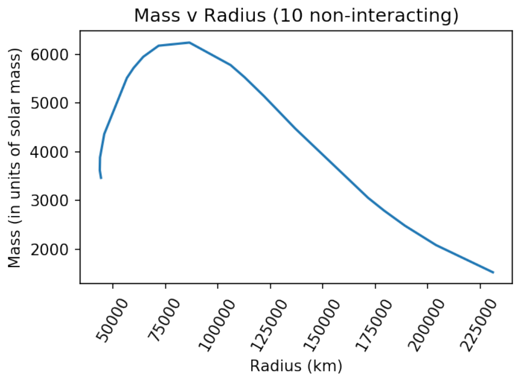
1. Input to the Program : Equation of State (EoS) (interacting, non-interacting for different fermion masses)
2. Starting condition : Defining Central Number Density, Pressure and Energy Density from a point in the EoS.
3. From the central density and pressure, the star's total mass and radius are calculated keeping in mind hydrostatic equilibrium, i.e., by solving the TOV equation and the mass continuity equation. *Numerical Methods used*: Fourth order Runge-Kutta method, Logarithmic interpolation.
4. The parameters of the star are iteratively calculated until we hit the cut-off surface pressure ($P = 10^{-23}\ MeV/fm^3$).
5. Program outputs total mass and radius for the given central density.
6. Steps 3-5 are repeated for the next starting conditions, chosen from the input EoS.

Thus, a family of stars is obtained for each EoS, with masses and radii parameterised by the central values.

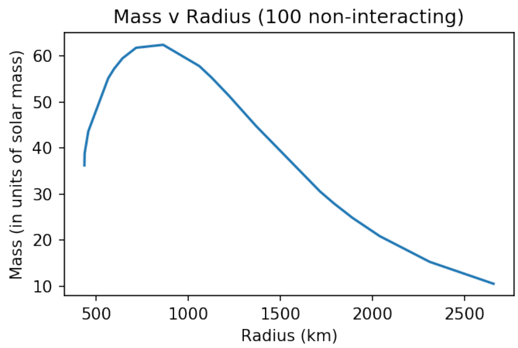
3.4.2 Compact Stars made of Non-Interacting, Completely Degenerate Fermi Gas

Compact stars made of free fermi gas are generated for different fermion masses, viz., m_f = 10 MeV, 100 MeV, 500 MeV, 1000 MeV. We would like to investigate the effects of fermion mass on the mass-radius relationship of compact stars.

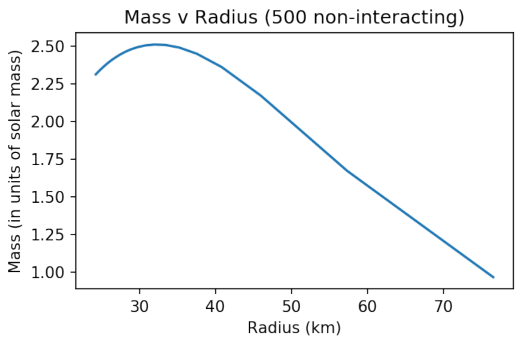
From fig (3.2), we see that each mass-radius curve follows the same kind of curve qualitatively, corresponding to a gravitationally bound system. Each curve peaks at a certain value. This is the maximum mass of the degenerate system. Beyond this, unphysical solutions are obtained.



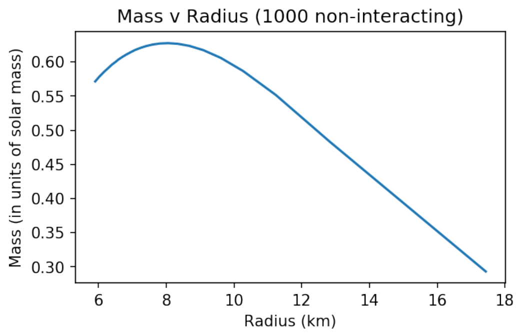
(a) 10 MeV



(b) 100 MeV



(c) 500 MeV



(d) 1000 MeV

Figure 3.2: Mass-Radius relationships for Compact Stars made of Non-Interacting Fermi Gas for various fermion masses

As the mass of the fermion increases, the degeneracy pressure at a given number density decreases. Qualitatively, the fixed number density fixes a length scale for the gas, which in turn fixes the momentum scale. At a fixed momentum, kinetic energy and thus pressure varies inversely with mass. Lower pressures lead to lower masses being supported at hydrostatic equilibrium. The maximum mass supported by the degenerate system decreases as the mass of fermion m_f increases.

m_f	Maximum Mass (M_{max})	Corresponding Radius (R_{crit})
10	6.244030695237450345e+03	8.638400001000000339e+04
100	6.243999425922786628e+01	8.638400099993593813e+02
500	2.512569296632380933e+00	3.198001000000220273e+01
1000	6.274992366402150079e-01	8.060009999999872221e+00

Table 3.1: M_{max} and R_{crit} for the different m_f values (non-interacting)

Table 3.1 shows the variation of the maximum mass and the corresponding radius with the fermion mass.

The exact relationship between the limiting mass and radius on m_f is seen in graphs (3.3) and (3.4) where M_{max} and R are plotted against $1/m_f^2$ to obtain straight lines. Thus,

$$M_{max} \propto 1/m_f^2 \qquad\qquad R_{crit} \propto 1/m_f^2 \tag{3.7}$$

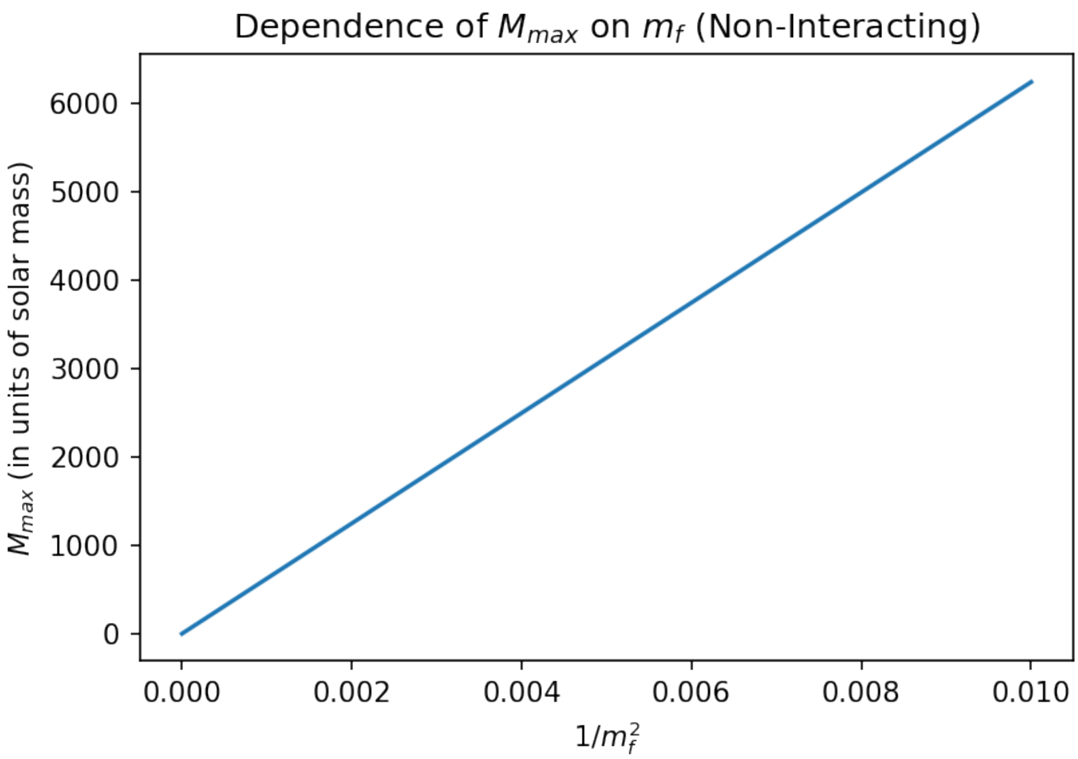


Figure 3.3: Dependence of M_{max} on m_f (Non-Interacting)

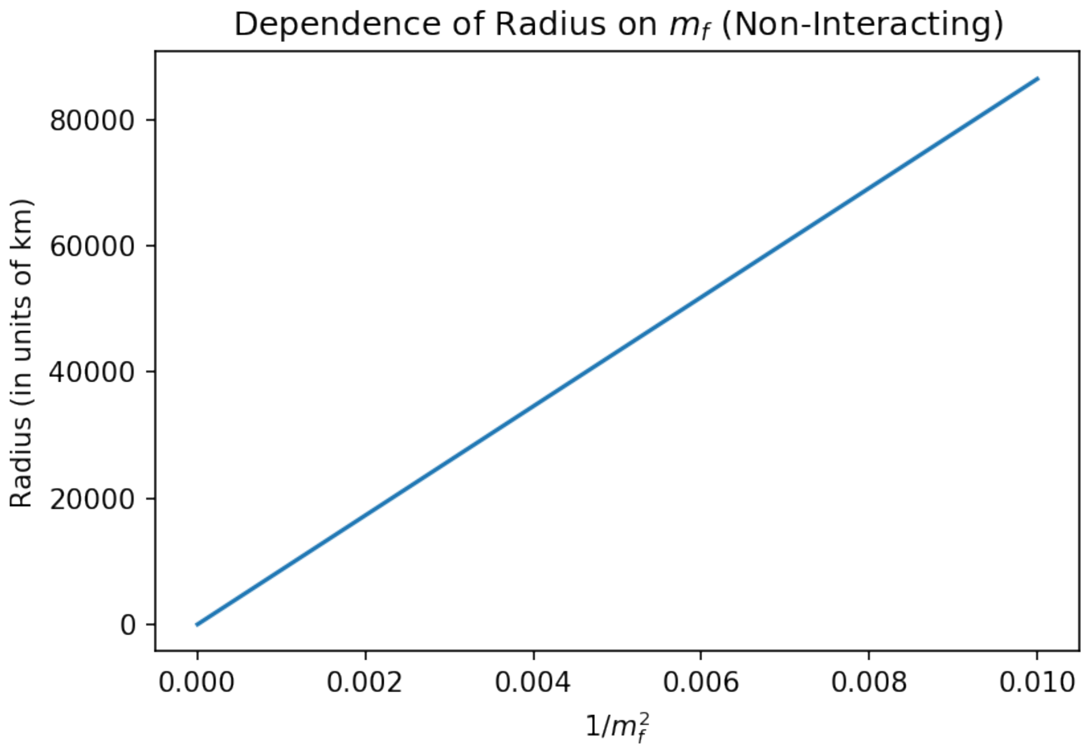


Figure 3.4: Dependence of R_{crit} on m_f (Non-Interacting)

3.4.3 Compact Stars made of Strongly Interacting, Completely Degenerate Fermi gas

Here, we consider completely degenerate relativistic fermi gas with strong two-body repulsive interactions. Let us introduce the interaction energy density to be of the form $\epsilon_{int} = cn^{\gamma}$, where c is the strength of interaction and γ is a constant. For strong, repulsive two-body interactions between fermions, we assume $\gamma = 2$. Then, $P_{int} \approx cn^2 \approx \epsilon_{int}$. This is stiffest possible EoS known as the Zeldovich EoS. It defines the limiting causal EoS, i.e, it gives the maximum possible pressure for a given energy density as allowed by causality [6].

In natural units, the constant c has units of mass^{-2}. Let $c = 1/m_i^2$, where m_i is the vector boson mediator mass. By introducing all the fundamental constants, and expressing the number density as function of the dimensionless fermi momentum (x_f) we arrive at the following expressions for interaction

$$P_{int} = \frac{1}{3\pi^2} \frac{x_f^6 (m_f c^2)^6}{\hbar c^3 (m_i c^2)^2} \tag{3.8}$$

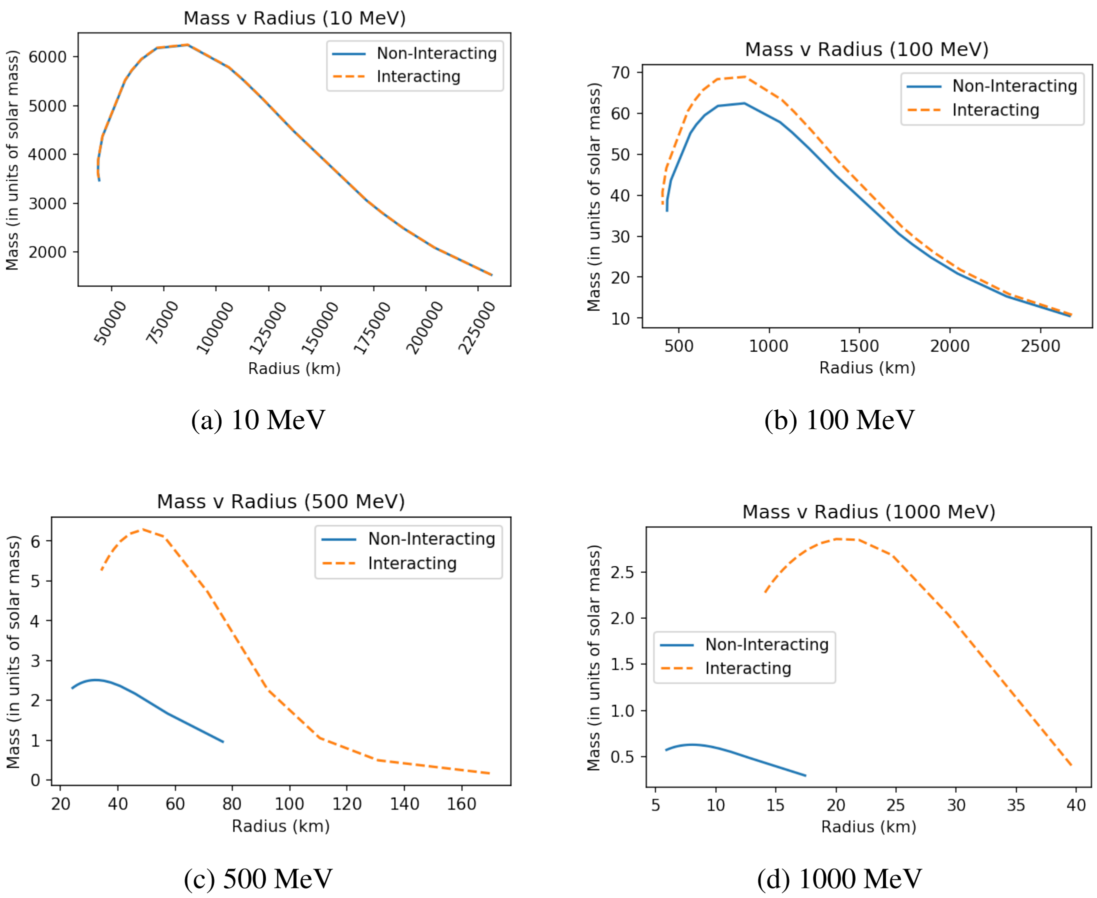
$$\epsilon_{int} = \frac{1}{3\pi^2} \frac{x_f^6 (m_f c^2)^6}{\hbar c^3 (m_i c^2)^2} \tag{3.9}$$

Then, the complete EoS for strongly interacting fermi gas is given by,

$$P_{tot} = P + P_{int} = \frac{\epsilon_0}{24}\left[(2x^3 - 3x)(1+x^2)^{1/2} + 3\sinh^{-1}(x)\right] + \frac{1}{3\pi^2}\frac{x_f^6 (m_f c^2)^6}{\hbar c^3 (m_i c^2)^2} \quad (3.10)$$

$$\epsilon_{tot} = \epsilon + \epsilon_{int} = \frac{\epsilon_0}{8}\left[(2x^3 + x)(1+x^2)^{1/2} - \sinh^{-1}(x)\right] + \frac{1}{3\pi^2}\frac{x_f^6 (m_f c^2)^6}{\hbar c^3 (m_i c^2)^2} \quad (3.11)$$

Let us assume $m_i = 100MeV$, typical of strong interaction. We would like to see the effects of interaction and fermion mass on the mass-radius relationship of non-rotating stars. In fig 3.5, plots of mass vs. radius for non-interacting and strongly interacting fermionic compact stars are plotted for four different fermion masses - m_f = 10 MeV, 100 MeV, 500 MeV and 1000 MeV.



(a) 10 MeV

(b) 100 MeV

(c) 500 MeV

(d) 1000 MeV

Figure 3.5: Mass-Radius relationship of non-interacting and strongly interacting fermionic compact stars for various fermion masses and fixed vector boson mass of 100 MeV

Let us introduce a dimensionless interaction strength $y = m_f/m_i$. Table (3.2) lists the maximum mass and corresponding radius for four values of m_f and y.

m_f	y	Maximum Mass (M_{max})	Corresponding Radius (R_{crit})
10	0.1	6.250659838329802369e+03	8.638600001000000339e+04
100	1	6.890253370484283835e+01	8.671100099993564072e+02
500	5	6.289929847785751704e+00	4.881000999999886147e+01
1000	10	2.857178782742042955e+00	2.002001000000033315e+01

Table 3.2: M_{max} and R_{crit} for the different m_f values (interacting)

We observe the following:

- Strongly interacting fermionic compact stars have higher masses compared to their non-interacting counterparts. This is because the strong repulsion pressure is much more effective in stabilising the star against gravitational collapse than the degeneracy pressure alone.
- In the 10 MeV case (corresponding to $y = 0.1$), the interacting and non-interacting cases are basically identical and overlap with each other.
- As the mass of the fermion m_f increases, the separation between the interacting and non-interacting cases becomes more and more prominent.
- The curves are maximally separated for the 1000 MeV case, where $y = 10$.

For lower fermion masses, i.e., $y < 1$ (or) $m_f < m_i$, interacting and non-interacting cases are basically identical. In case of lower fermion masses, there is weak interaction. The fermions become relativistic before the interaction terms dominate the pressure. The maximum mass is reached before interactions are important.

For higher fermion masses, i.e., $y > 1$ (or) $m_f > m_i$ interaction plays a significant role. In case of higher fermion masses, there is strong interaction. The interaction terms dominate the pressure terms before the relativistic limit is reached. The masses are able to reach higher values at the same radii [6].

To analyse the effects of mediator mass/interaction strength m_i on the mass-radius relationship, graphs are generated by keeping m_f constant at 500 MeV, while varying m_i as 5, 10, 50, 100, and 500 MeV, as shown in fig (3.6).

At constant m_f, as m_i decreases, y increases, i.e, interactions get stronger. As the repulsive force increases, the star is able to push back more and more against gravity. The star supports higher masses as the interactions get stronger, i.e, as m_i decreases.

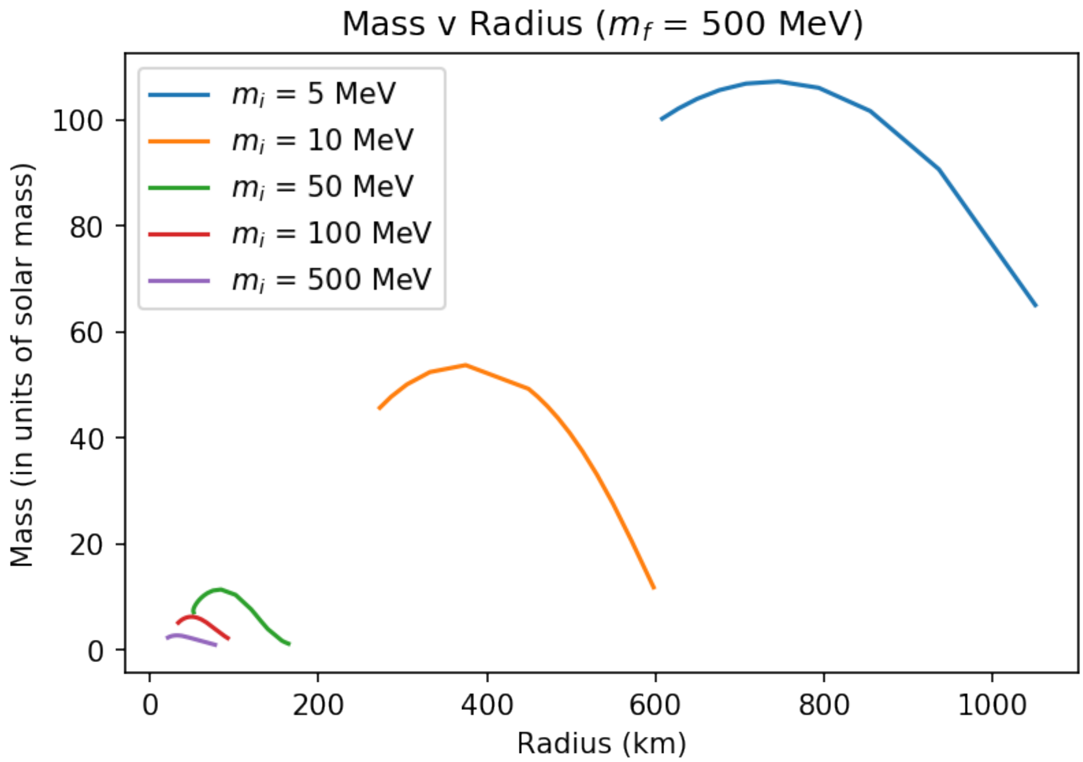


Figure 3.6: Mass-radius relationship of strongly interacting fermionic compact stars of fixed fermion mass of 500 MeV and various mediator masses.

m_i	Maximum Mass (M_{max})	Corresponding Radius (R_{crit})
5	1.072752861593063898e+02	7.455800099994669381e+02
10	5.376294922228562001e+01	3.746400099998042492e+02
50	1.143133951821683247e+01	8.440001000000627585e+01
100	6.289115307682870970e+00	4.999000999999862671e+01
500	2.770896884985470709e+00	3.352001000000190345e+01

Table 3.3: M_{max} and R_{crit} for the different m_i values (m_f = 500 MeV)

Table (3.3) gives the values of maximum mass M_{max} and the corresponding critical radius R_{crit} for varying m_i values for a fixed fermion mass, m_f = 500 MeV.

To get the exact dependence of the limiting mass and radius on m_i, we observe that the plot of M_{max} and R_{crit} against $1/m_i$ gives a straight line (figures (3.7) and (3.8)). This shows us that,

$$M_{max} \propto 1/m_i \qquad\qquad R_{crit} \propto 1/m_i \tag{3.12}$$

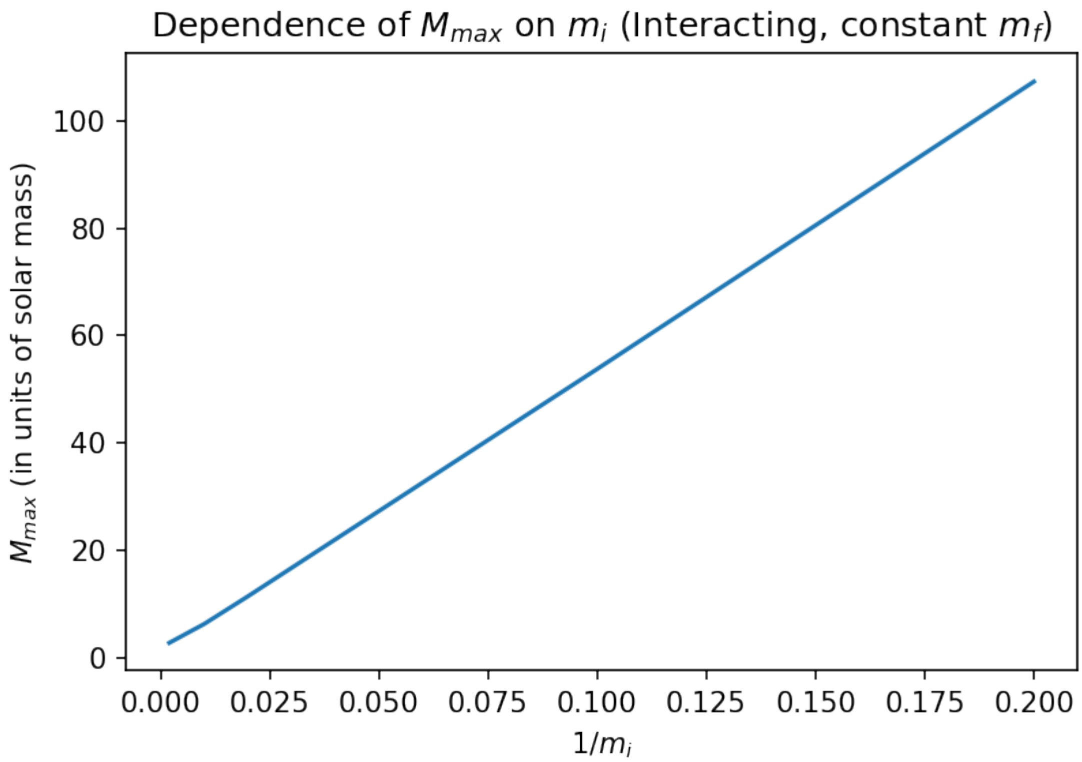


Figure 3.7: Dependence of M_{max} on m_i

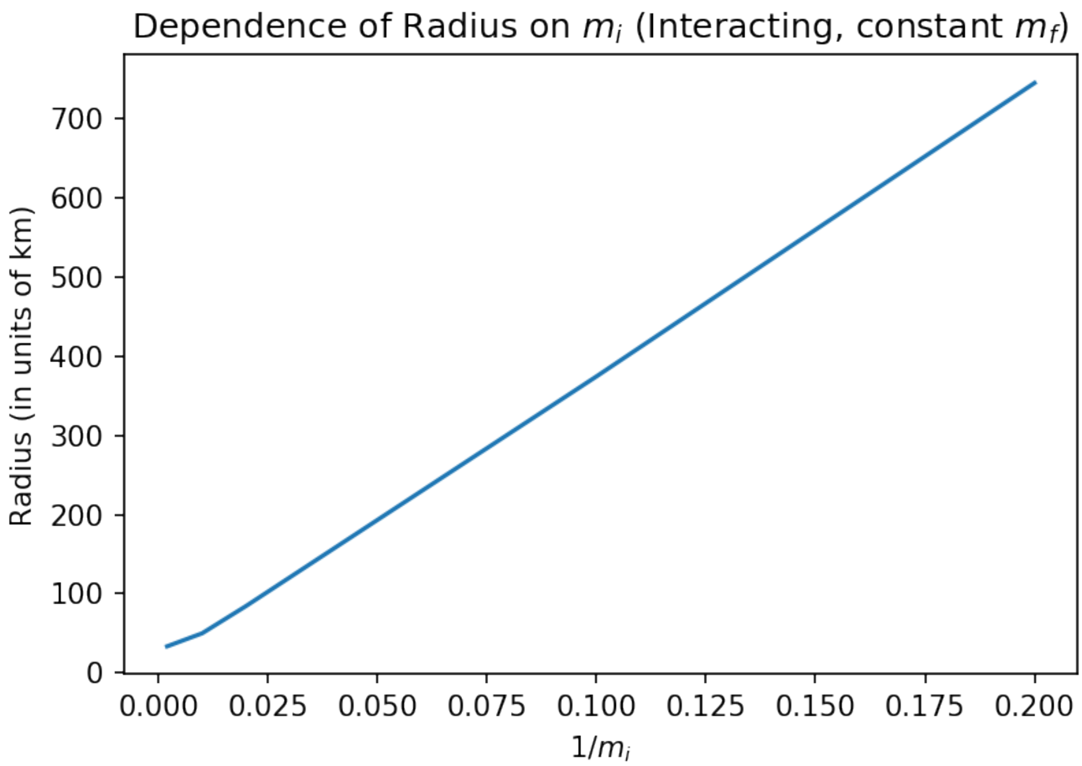


Figure 3.8: Dependence of R_{crit} on m_i

Chapter 4

SUMMARY AND CONCLUSIONS

Stellar hydrostatic equilibrium is the mechanism responsible for the stability of stars. It is the balance between the inward push of gravity and the outward push of pressure at every point in the star. As stars run out of fuel for nuclear burning, they collapse into objects of extremely high densities. These are Compact Stars - the end point of stellar evolution.

Stellar systems are described by the equations of stellar structure. This includes the equation of hydrostatic equilibirum along with the mass continuity equation. The equations of stellar structure are not generally analytically solvable unless we specify an equation of state to relate the pressure to the energy density of the star.

Assuming a polytropic equation of state ($P = K\epsilon^{\gamma}$) reduced the system to one differential equation. This gave us Lane-Emden equation - a dimensionless equation for the gravitational potential of a Newtonian, self-gravitating, spherically symmetric, polytropic fluid.. This equation was solved for different polytropic indices n - giving us stable and unstable branches for the radius of the star,

As compact stars are objects of extremely high density, General Relativistic corrections become important. Incorporating GR terms modifies the equation of hydrostatic equilibrium to the Tolman-Oppenheimer-Volkoff (TOV) equation. The Buchdahl limit was calculated - which gave us a lower bound on the radius of general relativistic, gravitationally bound objects before the central pressure diverges - given as $R > \frac{9GM}{4c^2}$. This limit holds for all realistic stars, i.e., with an energy density that decreases with radius.

In this project we investigated fermionic compact stars of different fermion masses, m_f. For each m_f, the relativistic, completely degenerate non-interacting and strongly interating EoS was studied. The pressure responsible to stabilise these stars against gravitational collapse is the Fermi degeneracy pressure, produced due to the Pauli exclusion principle. In the interacting case, the additional repulsive interactions also play a role in stabilisation, further stabilising the star against gravitational collapse.

There exists a limiting maximum mass (M_{max}) for every degenerate star beyond which it cannot be stabilised against gravitational collapse. This limiting mass is closely related with the mass of the fermion (m_f) that makes up the star. For a given number density, a higher fermion mass corresponds to a lower degeneracy pressure, and hence a lower mass supported by hydrostatic equilibrium.

The mass-radius relationship of fermionic compact stars, and thus the macroscopic quantities M_{max} and corresponding radius R_{crit}, are closely related to the microscopic m_f, and the relationship is given by

$$M_{max} \propto 1/m_f^2 \qquad\qquad R_{crit} \propto 1/m_f^2$$

In the strongly interacting case, repulsion provides an additional outward push, leading to higher masses being supported when compared to non-interacting cases. Interactions were introduced in the form of the Zeldovich equation of state, which is the stiffest causal EoS.

Interactions are mediated through particles of mass m_i. Weak interactions ($m_f << m_i$) leave the mass-radius relationship unchanged while strong interactions ($m_f >> m_i$) show higher masses being supported at the same radii.

By varying m_i while keeping m_f constant, the exact dependence of the mass-radius relationship on the interactions was analysed and found to be

$$M_{max} \propto 1/m_i \qquad\qquad R_{crit} \propto 1/m_i$$

Further investigations can be done into the role of more realistic interaction EoS on the mass-radius relationship, and thus the limiting mass (M_{max}) and corresponding radius (R_{crit}) of fermionic compact stars.

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